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Letter to the editor

Stability analysis on delayed neural networks based on an improved delay-partitioning approach

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ABSTRACT

In this paper, the asymptotical stability is investigated for a class of delayed neural networks (DNNs), in which one improved delay-partitioning idea is employed. By choosing an augmented Lyapunov–Krasovskii functional and utilizing general convex combination method, two novel conditions are obtained in terms of linear matrix inequalities (LMIs) and the conservatism can be greatly reduced by thinning the partitioning of delay intervals. Moreover, the LMI-based criteria heavily depend on both the upper and lower bounds on time-delay and its derivative, which is different from the existent ones. Though the results are not presented via standard LMIs, they still can be easily checked by resorting to Matlab LMI Toolbox. Finally, three numerical examples are given to demonstrate that our results can be less conservative than the present ones.

1. Introduction

Neural networks have been found many successful applications in various fields due to their strong capability of handling formidable problems and improving systems' performance. Presently, owing to the fact that in biological and artificial neural systems, there inevitably exist integration and communication delays which may induce oscillation, instability, or other poor performances, great efforts have been imposed on stability analysis on neural networks with time-delay, and many elegant results have been proposed in the relevant literature.

In recent years, because the Lyapunov–Krasovskii functional method can fully apply the information on time-delay of systems, the delay-dependent stability analysis has become an important topic of primary significance, in which the main purpose is to derive an allowable delay upper bound such that DNNs are asymptotically or exponentially stable [1–19]. Among those present approaches, the delay-partitioning idea has been proven to be more effective and constructive than the earlier ones, which was illustrated in [10–19]. Since Gu [20] initially proposed an effective delay-partitioning idea in 2001, many researchers have employed and further improved Gu's idea to analyze the stability of delayed systems including delayed neural networks [21–23,10]. Later, another novel delay-partitioning idea was put forward in [11] and achieved some developments [12,24–26,13–17], which was proven to be more concise and effective than the ones in [20–23,10]. Moreover, since the idea [11,12,24] cannot effectively deal with the variable delay, some researchers have improved the idea to analyze the stability of time-delayed systems including DNNs in [25,26,13–17], in which time-varying delays were addressed. However, there still exist some following points for the ideas in [25,26,13–17] waiting for further improvements. Firstly, these ideas cannot efficiently analyze the interval variable delay, especially as the lower bound of the delay is available precisely. Secondly, as for the variable delay, the delay-partitioning ideas have not fully employed the information on every

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subintervals of delay intervals, which could be illustrated by the constructions of the Lyapunov–Krasovskii functionals [25,26,13–17]. Thirdly, most works in [25,26,13–19] have not taken the lower bound of delay derivative into consideration. In fact, the available lower bound of delay derivative can play an important role in reducing conservatism of stability criteria derived in [27].

Inspired by the above discussion, in this paper, by making great efforts to improve the delay-partitioning idea described in [13–17], we investigate the asymptotical stability for neural networks with interval variable delay, in which the lower bound of delay derivative is involved. Together with one augmented Lyapunov–Krasovskii functional and generalized convex combination technique, two novel conditions are formulated in terms of LMIs and their feasibility can be easily checked with the help of Matlab LMI Toolbox. Finally, three numerical examples are given to illustrate that the proposed results in the paper are less conservative than the existing ones.

Notations: The symmetric term in a symmetric matrix is denoted by $*$, i.e., $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ * & Z \end{bmatrix}$.

2. Problem formulations

Consider the delayed neural networks described by the following form:

$$\dot{z}(t) = -b(z(t)) + Ag(z(t)) + Bg(z(t - \tau(t))) + L, \tag{1}$$

where $z = [z_1, \ldots, z_n]^T \in \mathbf{R}^n$ is a real n-vector denoting the state variables associated with the neurons, $b(z) = [b_1(z_1), \ldots, b_n(z_n)]^T$ is the behaved function, $g(z) = [g_1(z_1), \ldots, g_n(z_n)]^T$ represents the neuron activation function, $L = [l_1, \ldots, l_n]^T \in \mathbf{R}^n$ is a constant input vector, and A, B are the appropriately dimensional constant matrices.

The following assumptions on system (1) are made throughout this paper:

H1. Here, $\tau(t)$ denotes the interval time-varying delay satisfying

$$0 \le \tau_0 \le \tau(t) \le \tau_m, \quad \mu_0 \le \dot{\tau}(t) \le \mu_m, \tag{2}$$

and introduce $\bar{\tau}_m = \tau_m - \tau_0$, $\bar{\mu}_m = \mu_m - \mu_0$.

H2. Each function $\beta_i(\cdot) : \mathbf{R} \to \mathbf{R}$ is locally Lipschitz and there exist γ_i such that $\dot{\beta}_i(z) \ge \gamma_i > 0$ for all $z \in \mathbf{R}$. Here, we denote $\Gamma = \mathrm{diag}\{\gamma_1, \ldots, \gamma_n\}$.

H3. For the constants σ_j^+, σ_j^-, the bounded function $g_j(\cdot)$ in (1) satisfies the following condition

$$\sigma_j^- \le \frac{g_j(\alpha) - g_j(\beta)}{\alpha - \beta} \le \sigma_j^+, \quad \forall \alpha, \beta \in \mathbf{R},\ \alpha \ne \beta,\ j = 1, 2, \ldots, n,$$

and we introduce the denotations $\bar{\Sigma} = \mathrm{diag}\{\sigma_1^+, \ldots, \sigma_n^+\}$, $\Sigma = \mathrm{diag}\{\sigma_1^-, \ldots, \sigma_n^-\}$,

$$\Sigma_1 = \mathrm{diag}\{\sigma_1^+\sigma_1^-, \ldots, \sigma_n^+\sigma_n^-\}, \qquad \Sigma_2 = \mathrm{diag}\left\{\frac{\sigma_1^+ + \sigma_1^-}{2}, \ldots, \frac{\sigma_n^+ + \sigma_n^-}{2}\right\}. \tag{3}$$

It is clear that under H1–H3, system (1) has one equilibrium point $z^* = [z_1^*, \ldots, z_n^*]^T$. The equilibrium point of system (1) can be shifted to the origin by the transformation $x = z - z^*$, which converts system (1) to

$$\dot{x}(t) = -\beta(x(t)) + Af(x(t)) + Bf(x(t - \tau(t))), \tag{4}$$

where $x = [x_1, \ldots, x_n]^T$ is the state vector of transformed system (4), $\beta(x) = [\beta_1(x_1), \ldots, \beta_n(x_n)]^T$, $f(x) = [f_1(x_1), \ldots, f_n(x_n)]^T$; and $\beta_i(x_i) = b(x_i + z_i^*) - b(z_i^*)$, $f_i(x_i) = g_i(x_i + z_i^*) - g_i(z_i^*)$, $i = 1, 2, \ldots, n$. Note that the function $f_i(\cdot)$ satisfies $f_i(0) = 0$, and

$$\sigma_i^- \le \frac{f_i(\alpha)}{\alpha} \le \sigma_i^+, \quad \forall \alpha \in \mathbf{R},\ \alpha \ne 0,\ i = 1, 2, \ldots, n. \tag{5}$$

Then the problem to be addressed in the paper can be equivalently formulated as developing a condition ensuring that system (4) is asymptotically stable.

In order to obtain the stability criterion for system (4), the following lemma will be introduced.

Lemma 1 (*[28]*). *Suppose that Ω, Ξ_{1i}, Ξ_{2i} $(i = 1, 2)$ are the constant matrices of appropriate dimensions, $\alpha \in [0, 1]$, and $\beta \in [0, 1]$, then $\Omega + [\alpha\Xi_{11} + (1-\alpha)\Xi_{12}] + [\beta\Xi_{21} + (1-\beta)\Xi_{22}] < 0$ holds, if the following inequalities $\Omega + \Xi_{11} + \Xi_{21} < 0$, $\Omega + \Xi_{11} + \Xi_{22} < 0$, $\Omega + \Xi_{12} + \Xi_{21} < 0$, and $\Omega + \Xi_{12} + \Xi_{22} < 0$ hold simultaneously.*

3. Delay-dependent stability criteria

Firstly, we can represent system (4) as

$$\dot{x}(t) = y(t), \quad y(t) = -\beta(x(t)) + Af(x(t)) + Bf(x(t - \tau(t))). \tag{6}$$

For given positive integers m, l, then denoting $\varrho = \frac{\tau_0}{m}$, $\delta = \frac{\bar{\tau}_m}{l}$, $\rho(t) = \frac{\tau(t)-\tau_0}{l}$, and using assumptions H1–H3, we can construct the Lyapunov–Krasovskii functional:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)), \tag{7}$$

where

$$\begin{aligned}
V_1(x(t)) &= x^T(t)Px(t) + 2\sum_{i=1}^{n} q_i \int_0^{x_i} \left[\beta_i(s) - \gamma_i s\right] \mathrm{d}s + 2\sum_{i=1}^{n} k_i \int_0^{x_i} [f_i(s) - \sigma_i^- s] \mathrm{d}s + 2\sum_{i=1}^{n} f_i \int_0^{x_i} [\sigma_i^+ s - f_i(s)] \mathrm{d}s, \\
V_2(x(t)) &= \int_{t-\varrho}^{t} \begin{bmatrix} \gamma(s) \\ g(\gamma(s)) \end{bmatrix}^T \begin{bmatrix} P_1 & H_1 \\ * & Q_1 \end{bmatrix} \begin{bmatrix} \gamma(s) \\ g(\gamma(s)) \end{bmatrix} \mathrm{d}s + \int_{t-\tau_0-\delta}^{t-\tau_0} \begin{bmatrix} \sigma(s) \\ h(\sigma(s)) \end{bmatrix}^T \begin{bmatrix} P_2 & H_2 \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \sigma(s) \\ h(\sigma(s)) \end{bmatrix} \mathrm{d}s \\
&\quad + \sum_{i=1}^{l}\sum_{j=i}^{l} \int_{t-\tau_0-(j-1)\delta-\rho(t)}^{t-\tau_0-(i-1)\delta} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \begin{bmatrix} X_{1ij} & Y_{1ij} \\ * & Z_{1ij} \end{bmatrix} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} \mathrm{d}s \\
&\quad + \sum_{i=1}^{l}\sum_{j=i}^{l} \int_{t-\tau_0-j\delta}^{t-\tau_0-(i-1)\delta-\rho(t)} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \begin{bmatrix} X_{2ij} & Y_{2ij} \\ * & Z_{2ij} \end{bmatrix} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} \mathrm{d}s, \\
V_3(x(t)) &= \sum_{i=1}^{m} \int_{-i\varrho}^{-(i-1)\varrho} \int_{t+\theta}^{t} y^T(s) V_i y(s) \mathrm{d}s\mathrm{d}\theta + \sum_{i=1}^{l}\sum_{j=i}^{l} \int_{-\tau_0-j\delta}^{-\tau_0-(i-1)\delta} \int_{t+\theta}^{t} y^T(s) W_{ij} y(s) \mathrm{d}s\mathrm{d}\theta
\end{aligned}$$

with $Q = \mathrm{diag}\{q_1, \ldots, q_n\}$, $K = \mathrm{diag}\{k_1, \ldots, k_n\}$, $F = \mathrm{diag}\{f_1, \ldots, f_n\}$, $ln \times ln$ constant matrices P_1, Q_1, H_1, $mn \times mn$ constant matrices P_2, Q_2, H_2, $n \times n$ constant matrices $P, V_i, X_{1ij}, Y_{1ij}, Z_{1ij}, X_{2ij}, Y_{2ij}, Z_{2ij}, W_{ij}$, and

$$\begin{aligned}
\gamma^T(s) &= \begin{bmatrix} x^T(s) & \cdots & x^T(s-(m-1)\varrho) \end{bmatrix}, \qquad g^T(\gamma(s)) = \begin{bmatrix} f^T(x(s)) & \cdots & f^T(x(s-(m-1)\varrho)) \end{bmatrix}, \\
\sigma^T(s) &= \begin{bmatrix} x^T(s) & \cdots & x^T(s-(l-1)\delta) \end{bmatrix}, \qquad h^T(\sigma(s)) = \begin{bmatrix} f^T(x(s)) & \cdots & f^T(x(s-(l-1)\delta)) \end{bmatrix}.
\end{aligned}$$

Denoting a parameter set $\Phi = \left\{P, Q, K, F, V_d, \begin{bmatrix} P_g & H_g \\ * & Q_g \end{bmatrix}, \begin{bmatrix} X_{gij} & Y_{gij} \\ * & Z_{gij} \end{bmatrix}, W_{ij}, d = 1, \ldots, m; g = 1, 2; i, j = 1, \ldots, l\right\}$, then we give one proposition which is essential to the following proof.

Proposition 1. *If the parameter set Φ satisfies the following condition:*

$$\begin{aligned}
&P > 0, \quad Q > 0, \quad K > 0, \quad F > 0, \quad V_d > 0, \quad \begin{bmatrix} P_1 & H_1 \\ * & Q_1 \end{bmatrix} > 0, \quad \begin{bmatrix} P_2 & H_2 \\ * & Q_2 \end{bmatrix} > 0, \\
&\begin{bmatrix} X_{1ij} & Y_{1ij} \\ * & Z_{1ij} \end{bmatrix} > 0, \quad \begin{bmatrix} X_{2ij} & Y_{2ij} \\ * & Z_{2ij} \end{bmatrix} > 0, \quad W_{ij} > 0, \quad d = 1, \ldots, m; \quad g = 1, 2; \quad 1 \le i \le j \le l,
\end{aligned}$$

then the Lyapunov–Krasovskii functional (7) is definitely positive.

Moreover, in order to simplify the subsequent proof, we give some following denotations:

$$\bar{X}_h = \mathrm{diag}\left\{\sum_{i=1}^{l} X_{h1i}, \sum_{i=2}^{l} X_{h2i}, \ldots, X_{hll}\right\}, \qquad \bar{Y}_h = \mathrm{diag}\left\{\sum_{i=1}^{l} Y_{h1i}, \sum_{i=2}^{l} Y_{h2i}, \ldots, Y_{hll}\right\}, \quad h = 1, 2, \tag{8}$$

$$\bar{Z}_h = \mathrm{diag}\left\{\sum_{i=1}^{l} Z_{h1i}, \sum_{i=2}^{l} Z_{h2i}, \ldots, Z_{hll}\right\}, \qquad \tilde{X}_h = \mathrm{diag}\left\{X_{h11}, \sum_{i=1}^{2} X_{hi2}, \ldots, \sum_{i=1}^{l} X_{hil}\right\}, \quad h = 1, 2, \tag{9}$$

$$\tilde{Y}_h = \mathrm{diag}\left\{Y_{h11}, \sum_{i=1}^{2} Y_{hi2}, \ldots, \sum_{i=1}^{l} Y_{hil}\right\}, \qquad \tilde{Z}_h = \mathrm{diag}\left\{Z_{h11}, \sum_{i=1}^{2} Z_{hi2}, \ldots, \sum_{i=1}^{l} Z_{hil}\right\}, \quad h = 1, 2, \tag{10}$$

$$\tilde{T} = \mathrm{diag}\left\{T_1, \ldots, T_l\right\}, \qquad \tilde{R} = \mathrm{diag}\left\{R_1, \ldots, R_l\right\}. \tag{11}$$

In the next, based on the most improved techniques for achieving the criteria in [6,13–17,27,28], we state and establish the new delay-dependent stability criterion for system (4).

Theorem 1. *For given scalars $\tau_0, \tau_m, \mu_0, \mu_m$ in (2), and denoting $\pi = 2m + 4l + 6$, system (4) is globally asymptotically stable, if there exist one parameter set Φ satisfying Proposition 1, $n \times n$ matrices E_i $(i = 1, 2)$, $n \times n$ diagonal matrices $G > 0, U_i > 0$ $(i = 1, 2, 3, 4)$, $T_i > 0, R_i > 0$ $(i = 1, \ldots, l)$, and $\pi \times n$ constant matrices N_i $(i = 1, \ldots, m)$, M_{hj} $(h = 1, 2; j = 1, \ldots, l)$, H_{ij} $(1 \le i \le j \le l)$ such that the LMIs in (12)–(13) hold*

$$\begin{bmatrix} \Upsilon_1\Theta\Upsilon_1^T + \Upsilon_2\Xi\Upsilon_2^T + \$ + \$^T + \frac{\bar{\mu}_m}{l}\sum_{i=1}^{l}\sum_{j=i}^{l} I_{1j}^T \begin{bmatrix} X_{1ij} & Y_{1ij} \\ * & Z_{1ij} \end{bmatrix} I_{1j} & \Delta_1 & \Delta_{k2} & \Delta_3 \\ * & -\Omega_1 & 0 & 0 \\ * & * & -\Omega_2 & 0 \\ * & * & * & -\Omega_3 \end{bmatrix} < 0, \quad \forall k = 1, 2, \tag{12}$$

$$\begin{bmatrix} \Upsilon_1\Theta\Upsilon_1^T + \Upsilon_2\Xi\Upsilon_2^T + \$ + \$^T + \frac{\bar{\mu}_m}{l}\sum_{i=1}^{l}\sum_{j=i}^{l} I_{2i}^T \begin{bmatrix} X_{2ij} & Y_{2ij} \\ * & Z_{2ij} \end{bmatrix} I_{2i} & \Delta_1 & \Delta_{k2} & \Delta_3 \\ * & -\Omega_1 & 0 & 0 \\ * & * & -\Omega_2 & 0 \\ * & * & * & -\Omega_3 \end{bmatrix} < 0, \quad \forall k = 1, 2, \tag{13}$$

where $\Delta_1 = \sqrt{\varrho}\left[N_1, \ldots, N_m\right]$, $\Delta_{k2} = \sqrt{\delta}\left[M_{k1}, \ldots, M_{kl}\, M_{k2}, \ldots, M_{kl}, \ldots, M_{kl}\right]$, $\Delta_3 = \sqrt{\delta}\left[M_{12}, \ldots, M_{1l}, M_{23}, \ldots, M_{2l}, \ldots, M_{l-1,l}\right]$, $\Omega_1 = \left[V_1, \ldots, V_m\right]$, $\Omega_2 = \left[W_{11}, \ldots, W_{1l}\, W_{22}, \ldots, W_{2l}, \ldots, W_{ll}\right]$, $\Omega_3 = \left[W_{12}, \ldots, (l-1)W_{1l}\, W_{23}, \ldots, (l-2)W_{2l}, \ldots, W_{l-1,l}\right]$, $\$ = \left[N_1\ N_2 - N_1, \ldots, N_m - N_{m-1} - N_m + M_{21}\, 2M_{22} - M_{21}, \ldots, lM_{2l} - (l-1)M_{2(l-1)} - lM_{2l} 0_{\pi\cdot(m+l+1)n} M_{11} - M_{21} + \sum_{i=2}^{l} H_{1i}, \ldots, (l-1)[M_{1(l-1)} - M_{2(l-1)}] + H_{(l-1)l} - \sum_{i=1}^{l-2} H_{i(l-1)} l[M_{1l} - M_{2l}] - \sum_{i=1}^{l-1} H_{il} 0_{\pi\cdot(l+4)n}\right]$, $I_{1j} = \begin{bmatrix} 0_{n\cdot(2m+2l+1+j)n} - I_n\, 0_{n\cdot(l-1)n} - I_n * \\ 0_{n\cdot(2m+2l+1+j)n} - I_n\, 0_{n\cdot(l-1)n} - I_n * \end{bmatrix}$, $I_{2i} = \begin{bmatrix} 0_{n\cdot(2m+2l+1+i)n}\, I_n\, 0_{n\cdot(l-1)n}\, I_n * \\ 0_{n\cdot(2m+2l+1+i)n}\, I_n\, 0_{n\cdot(l-1)n}\, I_n * \end{bmatrix}$, *and*

$$\Theta = \begin{bmatrix} -2GF - U_1\Sigma_1 & 0 & 0 & \Theta_{14} & 0 & 0 & \Theta_{17} & 0 & E_1^T B & -E_1^T + G \\ * & -U_2\Sigma_1 & 0 & 0 & U_2\Sigma_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -U_3\Sigma_1 & 0 & 0 & U_3\Sigma_2 & 0 & 0 & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 & \Theta_{47} & 0 & 0 & A^T Q \\ * & * & * & * & -U_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -U_3 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Theta_{77} & 0 & E_2^T B & -E_2^T \\ * & * & * & * & * & * & * & -U_4\Sigma_1 & U_4\Sigma_2 & 0 \\ * & * & * & * & * & * & * & * & -U_4 & B^T Q \\ * & * & * & * & * & * & * & * & * & -Q - Q^T \end{bmatrix},$$

$$\Xi = \begin{bmatrix} P_1 - \tilde{T}\tilde{\Sigma}_1 & 0 & 0 & 0 & H_1 + \tilde{T}\tilde{\Sigma}_2 & 0 & 0 & 0 & 0 & 0 \\ * & -P_1 & 0 & 0 & 0 & -H_1 & 0 & 0 & 0 & 0 \\ * & * & P_2 + \bar{X}_1 & 0 & 0 & 0 & H_2 + \bar{Y}_1 & 0 & 0 & 0 \\ * & * & * & -P_2 - \tilde{X}_2 & 0 & 0 & 0 & -H_2 - \tilde{Y}_2 & 0 & 0 \\ * & * & * & * & Q_1 - \tilde{T} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & Q_2 + \bar{Z}_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -Q_2 - \tilde{Z}_2 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & \Xi_{9,10} \\ * & * & * & * & * & * & * & * & * & \Xi_{10,10} \end{bmatrix},$$

$$\Upsilon_1 = \begin{bmatrix} I_n & & * \\ 0_{n\cdot mn} & I_n & * \\ 0_{n\cdot(m+l)n} & I_n & * \\ 0_{n\cdot(m+l+1)n} & I_n & * \\ 0_{n\cdot(2m+l+1)n} & I_n & * \\ 0_{n\cdot(2m+2l+1)n} & I_n & * \\ 0_{n\cdot(2m+4l+2)n} & I_n & * \\ 0_{n\cdot(2m+4l+3)n} & I_n & * \\ 0_{n\cdot(2m+4l+4)n} & I_n & * \\ 0_{n\cdot(2m+4l+5)n} & I_n & \end{bmatrix}, \qquad \Upsilon_2 = \begin{bmatrix} I_{mn} & & * \\ 0_{mn\cdot n} & I_{mn} & * \\ 0_{ln\cdot mn} & I_{ln} & * \\ 0_{ln\cdot(m+1)n} & I_{ln} & * \\ 0_{mn\cdot(m+l+1)n} & I_{mn} & * \\ 0_{mn\cdot(m+l+2)n} & I_{mn} & * \\ 0_{ln\cdot(2m+l+1)n} & I_{ln} & * \\ 0_{ln\cdot(2m+l+2)n} & I_{ln} & * \\ 0_{ln\cdot(2m+2l+2)n} & I_{ln} & * \\ 0_{ln\cdot(2m+3l+2)n} & I_{ln} & * \end{bmatrix}$$

with all $*$ *above representing the appropriately dimensional* **0** *matrix guaranteeing* $I_{1j}, I_{2i}, \Upsilon_1, \Upsilon_2$ *of* π *columns, and*

$$\Theta_{14} = E_1^T A + U_1\Sigma_2, \qquad \Theta_{17} = P - \Gamma^T Q - \Sigma K + \bar{\Sigma} F - E_1^T, \qquad \Theta_{47} = K - F + A^T E_2,$$

$$\Theta_{77} = -E_2^T - E_2 + \sum_{i=1}^{m} \varrho V_i + \sum_{i=1}^{l}\sum_{j=i}^{l} \delta W_{ij}, \qquad \Xi_{99} = \left(1 - \frac{\mu_m}{l}\right)\bar{X}_2 - \left(1 - \frac{\mu_0}{l}\right)\tilde{X}_1 - \tilde{R}\tilde{\Sigma}_1,$$

$$\Xi_{9,10} = \left(1 - \frac{\mu_m}{l}\right)\bar{Y}_2 - \left(1 - \frac{\mu_0}{l}\right)\tilde{Y}_1 - \tilde{R}\tilde{\Sigma}_2, \qquad \Xi_{10,10} = \left(1 - \frac{\mu_m}{l}\right)\bar{Z}_2 - \left(1 - \frac{\mu_0}{l}\right)\tilde{Z}_1 - \tilde{R}.$$

Proof. Based on the Lyapunov–Krasovskii functional in (7), computing the derivative of $V_1(x(t))$ along system (6) and using any $n \times n$ constant matrices E_i $(i = 1, 2)$, we can obtain

$$\begin{aligned}\dot{V}_1(x(t)) =\ & 2x^T(t)Py(t) + 2\beta^T(x(t))Q\Big[-\beta(x(t)) + Af(x(t)) + Bf(x(t-\tau(t)))\Big] - 2x^T(t)\Gamma^T Qy(t)\\ & + 2[f(x(t)) - \Sigma x(t)]^T Ky(t) + 2[\bar{\Sigma}x(t) - f(x(t))]^T Fy(t)\\ & + 2\big[x^T(t)E_1^T + y^T(t)E_2^T\big]\big[-y(t) - \beta(x(t)) + Af(x(t)) + Bf(x(t-\tau(t)))\big].\end{aligned} \tag{14}$$

Together with the denotations in (8)–(10), the time derivative $\dot{V}_i(x(t))$ $(i = 2, 3)$ can be deduced as follows:

$$\begin{aligned}\dot{V}_2(x(t)) =\ & \Big[\gamma^T(t)P_1\gamma(t) + 2\gamma^T(t)H_1g(\gamma(t)) + g^T(\gamma(t))Q_1g(\gamma(t))\Big] - \Big[\gamma^T(t-\tau_0)P_1\gamma(t-\tau_0)\\ & + 2\gamma^T(t-\tau_0)H_1g(\gamma(t-\tau_0)) + g^T(\gamma(t-\tau_0))Q_1g(\gamma(t-\tau_0))\Big] + \Big[\sigma^T(t-\tau_0)P_2\sigma(t-\tau_0)\\ & + 2\sigma^T(t-\tau_0)H_2h(\sigma(t-\tau_0)) + h^T(\sigma(t-\tau_0))Q_2h(\sigma(t-\tau_0))\Big] - \Big[\sigma^T(t-\tau_0-\delta)\\ & \times P_2\sigma(t-\tau_0-\delta) + 2\sigma^T(t-\tau_0-\delta)H_2h(\sigma(t-\tau_0-\delta)) + h^T(\sigma(t-\tau_0-\delta))Q_2h(\sigma(t-\tau_0-\delta))\Big]\\ & + \sum_{i=1}^{l}\sum_{j=i}^{l}\Bigg\{\begin{bmatrix} x(t-\tau_0-(i-1)\delta) \\ f(x(t-\tau_0-(i-1)\delta))\end{bmatrix}^T \begin{bmatrix} X_{1ij} & Y_{1ij} \\ * & Z_{1ij}\end{bmatrix}\begin{bmatrix} x(t-\tau_0-(i-1)\delta) \\ f(x(t-\tau_0-(i-1)\delta))\end{bmatrix}\\ & - \left(1-\frac{\dot{\tau}(t)}{l}\right)\begin{bmatrix} x(t-\tau_0-(j-1)\delta-\rho(t)) \\ f(x(t-\tau_0-(j-1)\delta-\rho(t)))\end{bmatrix}^T \begin{bmatrix} X_{1ij} & Y_{1ij} \\ * & Z_{1ij}\end{bmatrix}\begin{bmatrix} x(t-\tau_0-(j-1)\delta-\rho(t)) \\ f(x(t-\tau_0-(j-1)\delta-\rho(t)))\end{bmatrix}\\ & + \left(1-\frac{\dot{\tau}(t)}{l}\right)\begin{bmatrix} x(t-\tau_0-(i-1)\delta-\rho(t)) \\ f(x(t-\tau_0-(i-1)\delta-\rho(t)))\end{bmatrix}^T \begin{bmatrix} X_{2ij} & Y_{2ij} \\ * & Z_{2ij}\end{bmatrix}\begin{bmatrix} x(t-\tau_0-(i-1)\delta-\rho(t)) \\ f(x(t-\tau_0-(i-1)\delta-\rho(t)))\end{bmatrix}\\ & - \begin{bmatrix} x(t-\tau_0-j\delta) \\ f(x(t-\tau_0-j\delta))\end{bmatrix}^T \begin{bmatrix} X_{2ij} & Y_{2ij} \\ * & Z_{2ij}\end{bmatrix}\begin{bmatrix} x(t-\tau_0-j\delta) \\ f(x(t-\tau_0-j\delta))\end{bmatrix}\Bigg\}\\ =\ & \Big[\gamma^T(t)P_1\gamma(t) + 2\gamma^T(t)H_1g(\gamma(t)) + g^T(\gamma(t))Q_1g(\gamma(t))\Big] - \Big[\gamma^T(t-\tau_0)P_1\gamma(t-\tau_0)\\ & + 2\gamma^T(t-\tau_0)H_1g(\gamma(t-\tau_0)) + g^T(\gamma(t-\tau_0))Q_1g(\gamma(t-\tau_0))\Big] + \Big[\sigma^T(t-\tau_0)P_2\sigma(t-\tau_0)\\ & + 2\sigma^T(t-\tau_0)H_2h(\sigma(t-\tau_0)) + h^T(\sigma(t-\tau_0))Q_2h(\sigma(t-\tau_0))\Big]\\ & - \Big[\sigma^T(t-\tau_0-\delta)P_2\sigma(t-\tau_0-\delta) + 2\sigma^T(t-\tau_0-\delta)H_2h(\sigma(t-\tau_0-\delta))\\ & + h^T(\sigma(t-\tau_0-\delta))Q_2h(\sigma(t-\tau_0-\delta))\Big] + \Big[\sigma^T(t-\tau_0)\bar{X}_1\sigma(t-\tau_0) + \sigma^T\big(t-\tau_0-\rho(t)\big)\\ & \times \left(1-\frac{\dot{\tau}(t)}{l}\right)(\bar{X}_2-\tilde{X}_1)\sigma\big(t-\tau_0-\rho(t)\big) - \sigma^T(t-\tau_0-\delta)\tilde{X}_2\sigma(t-\tau_0-\delta)\Big]\\ & + \Big[2\sigma^T(t-\tau_0)\bar{Y}_1h(\sigma(t-\tau_0)) + 2\sigma^T\big(t-\tau_0-\rho(t)\big)\left(1-\frac{\dot{\tau}(t)}{l}\right)(\bar{Y}_2-\tilde{Y}_1)h\big(\sigma(t-\tau_0-\rho(t))\big)\\ & - 2\sigma^T(t-\tau_0-\delta)\tilde{Y}_2h(\sigma(t-\tau_0-\delta))\Big] + \Big[h^T(\sigma(t-\tau_0))\bar{Z}_1h(\sigma(t-\tau_0)) + h^T\big(\sigma(t-\tau_0-\rho(t))\big)\\ & \times \left(1-\frac{\dot{\tau}(t)}{l}\right)(\bar{Z}_2-\tilde{Z}_1)h\big(\sigma(t-\tau_0-\rho(t))\big) - h^T\big(\sigma(t-\tau_0-\delta)\big)\tilde{Z}_2h^T\big(\sigma(t-\tau_0-\delta)\big)\Big],\end{aligned} \tag{15}$$

$$\begin{aligned}\dot{V}_3(x(t)) =\ & \sum_{i=1}^{m} y^T(t)\varrho V_iy(t) - \sum_{i=1}^{m}\int_{t-i\varrho}^{t-(i-1)\varrho} y^T(s)V_iy(s)\mathrm{d}s + \sum_{i=1}^{l}\sum_{j=i}^{l} y^T(t)\delta W_{ij}y(t)\\ & - \sum_{i=1}^{l}\sum_{j=i}^{l}\Big[\int_{t-\tau_0-j\delta}^{t-\tau_0-(j-1)\delta-\rho(t)} + \int_{t-\tau_0-(j-1)\delta-\rho(t)}^{t-\tau_0-(i-1)\delta-\rho(t)} + \int_{t-\tau_0-(i-1)\delta-\rho(t)}^{t-\tau_0-(i-1)\delta}\Big]y^T(s)W_{ij}y(s)\mathrm{d}s.\end{aligned} \tag{16}$$

By utilizing assumption H2, for any $n \times n$ diagonal matrix $G \ge 0$, we can easily derive

$$0 \le 2\big[x^T(t)G\beta(x(t)) - x^T(t)G\Gamma x(t)\big]. \tag{17}$$

From (5), the following inequality holds for any $n \times n$ diagonal matrices $U_i > 0\ (i = 1, 2, 3, 4)$, $T_i > 0$, $R_i > 0\ (i = 1, \ldots, l)$ and the denotations $\tilde{T}$, $\tilde{R}$ in (11),

$$\begin{aligned}
0 \le{} & \Big[-x^T(t)U_1\Sigma_1 x(t) + 2x^T(t)U_1\Sigma_2 f(x(t)) - f^T(x(t))U_1 f(x(t))\Big] + \Big[-x^T(t-\tau_0)U_2\Sigma_1 x(t-\tau_0) \\
& + 2x^T(t-\tau_0)U_2\Sigma_2 f(x(t-\tau_0)) - f^T(x(t-\tau_0))U_2 f(x(t-\tau_0))\Big] \\
& + \Big[-x^T(t-\tau_m)U_3\Sigma_1 x(t-\tau_m) + 2x^T(t-\tau_m)U_2\Sigma_2 f(x(t-\tau_m)) - f^T(x(t-\tau_m))U_3 f(x(t-\tau_m))\Big] \\
& + \Big[-x^T(t-\tau(t))U_4\Sigma_1 x(t-\tau(t)) + 2x^T(t-\tau(t))U_4\Sigma_2 f(x(t-\tau(t))) - f^T(x(t-\tau(t)))U_4 f(x(t-\tau(t)))\Big] \\
& + \Big[-\sigma^T(t-\tau_0)\tilde{T}\tilde{\Sigma}_1\sigma(t-\tau_0) + 2\sigma^T(t-\tau_0)\tilde{T}\tilde{\Sigma}_2 h(\sigma(t-\tau_0)) \\
& - h^T(\sigma(t-\tau_0))\tilde{T}h(\sigma(t-\tau_0))\Big] + \Big[-\sigma^T\big(t-\tau_0-\rho(t)\big)\tilde{R}\tilde{\Sigma}_1\sigma\big(t-\tau_0-\rho(t)\big) \\
& + 2\sigma^T\big(t-\tau_0-\rho(t)\big)\tilde{R}\tilde{\Sigma}_2 h\big(\sigma(t-\tau_0-\rho(t))\big) - h^T\big(\sigma(t-\tau_0-\rho(t))\big)\tilde{R}h\big(\sigma(t-\tau_0-\rho(t))\big)\Big].
\end{aligned} \tag{18}$$

Furthermore, we respectively denote Φ_i, Ψ_j, Π_{ij}, and Λ_i as follows:

$$\begin{aligned}
\Phi_i &= x\big(t-(i-1)\varrho\big) - x\big(t-i\varrho\big) - \int_{t-i\varrho}^{t-(i-1)\varrho} y(s)\mathrm{d}s, \quad 1 \le i \le m, \\
\Psi_j &= x\big(t-\tau_0-(j-1)\delta-\rho(t)\big) - x\big(t-\tau_0-j\delta\big) - \int_{t-\tau_0-j\delta}^{t-\tau_0-(j-1)\delta-\rho(t)} y(s)\mathrm{d}s, \quad 1 \le j \le l, \\
\Pi_{ij} &= x\big(t-\tau_0-(i-1)\delta-\rho(t)\big) - x\big(t-\tau_0-(j-1)\delta-\rho(t)\big) - \int_{t-\tau_0-(j-1)\delta-\rho(t)}^{t-\tau_0-(i-1)\delta-\rho(t)} y(s)\mathrm{d}s, \quad 1 \le i \le j \le l, \\
\Lambda_i &= x\big(t-\tau_0-(i-1)\delta\big) - x\big(t-\tau_0-(i-1)\delta-\rho(t)\big) - \int_{t-\tau_0-(i-1)\delta-\rho(t)}^{t-\tau_0-(i-1)\delta} y(s)\mathrm{d}s, \quad 1 \le i \le l.
\end{aligned}$$

By using any $\pi \times n$ constant matrices $N_i\ (i = 1, \ldots, m)$, $M_{hj}\ (h = 1, 2; j = 1, \ldots, l)$, $H_{ij}\ (1 \le i \le j \le l)$, we can obtain

$$2\zeta^T(t)\sum_{i=1}^{m} N_i\Phi_i + 2\zeta^T(t)\sum_{i=1}^{l}\sum_{j=i}^{l} M_{1j}\Psi_j + 2\zeta^T(t)\sum_{i=1}^{l}\sum_{j=i}^{l} H_{ij}\Pi_{ij} + 2\zeta^T(t)\sum_{i=1}^{l}\sum_{j=i}^{l} M_{2j}\Lambda_i = 0, \tag{19}$$

in which

$$\begin{aligned}
\zeta^T(t) ={} & \Big[\gamma^T(t)\ \sigma^T(t-\tau_0)\ x^T(t-\tau_m) g^T(\gamma(t))\ h^T(\sigma(t-\tau_0))\ f^T(x(t-\tau_m))\ \sigma^T(t-\tau_0-\rho(t)) \\
& h^T\big(\sigma(t-\tau_0-\rho(t))\big)\ y^T(t) x^T(t-\tau(t))\ f^T(x(t-\tau(t)))\ \beta^T(x(t))\Big].
\end{aligned}$$

Now, combining the terms (14)–(19) yields that $\dot{V}(x(t))$ satisfies

$$\begin{aligned}
\dot{V}(x(t)) \le{} & \zeta^T(t)\Bigg\{\Upsilon_1\Theta\Upsilon_1^T + \Upsilon_2\Xi\Upsilon_2^T + \$^T + \$ + \sum_{i=1}^{m}\varrho N_i V_i^{-1}N_i^T + \sum_{i=1}^{l}\sum_{j=i}^{l}(j-i)\delta H_{ij}W_{ij}^{-1}H_{ij}^T \\
& + (\delta-\rho(t))\sum_{i=1}^{l}\sum_{j=i}^{l} M_{1j}W_{ij}^{-1}M_{1j}^T + \rho(t)\sum_{i=1}^{l}\sum_{j=i}^{l} M_{2j}W_{ij}^{-1}M_{2j}^T + \frac{\dot{\tau}(t)-\mu_0}{l}\sum_{i=1}^{l}\sum_{j=i}^{l} I_{1j}^T\begin{bmatrix} X_{1ij} & Y_{1ij} \\ * & Z_{1ij}\end{bmatrix} I_{1j} \\
& + \frac{\mu_m-\dot{\tau}(t)}{l}\sum_{i=1}^{l}\sum_{j=i}^{l} I_{2i}^T\begin{bmatrix} X_{2ij} & Y_{2ij} \\ * & Z_{2ij}\end{bmatrix} I_{2i}\Bigg\}\zeta(t) := \zeta^T(t)\Omega(t)\zeta(t),
\end{aligned} \tag{20}$$

in which $\Theta, \Xi, \Upsilon_1, \Upsilon_2, \$, I_{1j}, I_{2i}$ are presented in (12)–(13). Then together with Lemma 1 and definition of Schur complement, the LMIs in (12)–(13) can guarantee $\Omega(t) < 0$, which indicates that the dynamics of system (4) is asymptotically stable according to the Lyapunov–Krasovskii stability theorem. It completes the proof.

Based on Theorem 1 and $C = \mathrm{diag}\{c_1, \ldots, c_n\} > 0$, if we choose the Lyapunov–Krasovskii functional as (7) and set $2\sum_{i=1}^{n} q_i\int_0^{x_i}\big[\beta_i(s) - \gamma_i s\big]\mathrm{d}s = 0$ in $V_1(x(t))$, we can derive one novel stability criterion for the system

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))). \quad \square \tag{21}$$

Theorem 2. *For given scalars τ_0, τ_m, μ_0, μ_m in (2), and denoting $\pi = 2m+4l+5$, system (21) is globally asymptotically stable, if there exist one parameter set Φ with $Q = 0$ satisfying Proposition 1, $n \times n$ matrices $E_i\ (i = 1, 2)$, $n \times n$ diagonal matrices*

$U_i > 0\,(i = 1, 2, 3, 4)$, $T_i > 0$, $R_i > 0\,(i = 1, \dots, l)$, and $\pi \times n$ constant matrices $N_i\,(i = 1, \dots, m)$, $M_{hj}\,(h = 1, 2; j = 1, \dots, l)$, $H_{ij}\,(1 \le i \le j \le l)$ such that the LMIs in (22)–(23) hold

$$\begin{bmatrix} \Upsilon_1 \Theta \Upsilon_1^T + \Upsilon_2 \Xi \Upsilon_2^T + \$ + \$^T + \dfrac{\bar{\mu}_m}{l} \sum_{i=1}^{l} \sum_{j=i}^{l} I_{1j}^T \begin{bmatrix} X_{1ij} & Y_{1ij} \\ * & Z_{1ij} \end{bmatrix} I_{1j} & \Delta_1 & \Delta_{k2} & \Delta_3 \\ * & -\Omega_1 & 0 & 0 \\ * & * & -\Omega_2 & 0 \\ * & * & * & -\Omega_3 \end{bmatrix} < 0, \quad \forall k = 1, 2, \tag{22}$$

$$\begin{bmatrix} \Upsilon_1 \Theta \Upsilon_1^T + \Upsilon_2 \Xi \Upsilon_2^T + \$ + \$^T + \dfrac{\bar{\mu}_m}{l} \sum_{i=1}^{l} \sum_{j=i}^{l} I_{2i}^T \begin{bmatrix} X_{2ij} & Y_{2ij} \\ * & Z_{2ij} \end{bmatrix} I_{2i} & \Delta_1 & \Delta_{k2} & \Delta_3 \\ * & -\Omega_1 & 0 & 0 \\ * & * & -\Omega_2 & 0 \\ * & * & * & -\Omega_3 \end{bmatrix} < 0, \quad \forall k = 1, 2, \tag{23}$$

where $\Delta_1 = \sqrt{\varrho}\big[N_1, \dots, N_m\big]$, $\Delta_{k2} = \sqrt{\delta}\big[M_{k1}, \dots, M_{kl}\, M_{k2}, \dots, M_{kl}, \dots, M_{kl}\big]$, $\Delta_3 = \sqrt{\delta}\big[M_{12}, \dots, M_{1l}, M_{23}, \dots, M_{2l}, \dots, M_{l-1,l}\big]$, $\Omega_1 = \big[V_1, \dots, V_m\big]$, $\Omega_2 = \big[W_{11}, \dots, W_{1l}\, W_{22}, \dots, W_{2l}, \dots, W_{ll}\big]$, $\Omega_3 = \big[W_{12}, \dots, (l-1)W_{1l}\, W_{23}, \dots, (l-2)W_{2l}, \dots, W_{l-1,l}\big]$, $\$ = \big[N_1\, N_2 - N_1, \dots, N_m - N_{m-1} - N_m + M_{21}\, 2M_{22} - M_{21}, \dots, lM_{2l} - (l-1)M_{2(l-1)} - lM_{2l} 0_{\pi\cdot(m+l+1)n} M_{11} - M_{21} + \sum_{i=2}^{l} H_{1i}, \dots, (l-1)[M_{1(l-1)} - M_{2(l-1)}] + H_{(l-1)l} - \sum_{i=1}^{l-2} H_{i(l-1)} l[M_{1l} - M_{2l}] - \sum_{i=1}^{l-1} H_{il} 0_{\pi\cdot(l+4)n}\big]$, $I_{1j} = \begin{bmatrix} 0_{n\cdot(2m+2l+1+j)n} - I_n\, 0_{n\cdot(l-1)n} - I_n\, * \\ 0_{n\cdot(2m+2l+1+j)n} - I_n\, 0_{n\cdot(l-1)n} - I_n\, * \end{bmatrix}$, $I_{2i} = \begin{bmatrix} 0_{n\cdot(2m+2l+1+i)n}\, I_n\, 0_{n\cdot(l-1)n}\, I_n\, * \\ 0_{n\cdot(2m+2l+1+i)n}\, I_n\, 0_{n\cdot(l-1)n}\, I_n\, * \end{bmatrix}$, and

$$\Theta = \begin{bmatrix} \Theta_{11} & 0 & 0 & \Theta_{14} & 0 & 0 & \Theta_{17} & 0 & E_1^T B \\ * & -U_2\Sigma_1 & 0 & 0 & U_2\Sigma_2 & 0 & 0 & 0 & 0 \\ * & * & -U_3\Sigma_1 & 0 & 0 & U_3\Sigma_2 & 0 & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 & \Theta_{47} & 0 & 0 \\ * & * & * & * & -U_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -U_3 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Theta_{77} & 0 & E_2^T B \\ * & * & * & * & * & * & * & -U_4\Sigma_1 & U_4\Sigma_2 \\ * & * & * & * & * & * & * & * & -U_4 \end{bmatrix},$$

$$\Xi = \begin{bmatrix} P_1 - \tilde{T}\tilde{\Sigma}_1 & 0 & 0 & 0 & H_1 + \tilde{T}\tilde{\Sigma}_2 & 0 & 0 & 0 & 0 & 0 \\ * & -P_1 & 0 & 0 & 0 & -H_1 & 0 & 0 & 0 & 0 \\ * & * & P_2 + \bar{X}_1 & 0 & 0 & 0 & H_2 + \bar{Y}_1 & 0 & 0 & 0 \\ * & * & * & -P_2 - \tilde{X}_2 & 0 & 0 & 0 & -H_2 - \tilde{Y}_2 & 0 & 0 \\ * & * & * & * & Q_1 - \tilde{T} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & Q_2 + \bar{Z}_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -Q_2 - \tilde{Z}_2 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & \Xi_{9,10} \\ * & * & * & * & * & * & * & * & * & \Xi_{10,10} \end{bmatrix},$$

$$\Upsilon_1 = \begin{bmatrix} I_n & & * \\ 0_{n\cdot mn} & I_n & * \\ 0_{n\cdot(m+l)n} & I_n & * \\ 0_{n\cdot(m+l+1)n} & I_n & * \\ 0_{n\cdot(2m+l+1)n} & I_n & * \\ 0_{n\cdot(2m+2l+1)n} & I_n & * \\ 0_{n\cdot(2m+4l+2)n} & I_n & * \\ 0_{n\cdot(2m+4l+3)n} & I_n & * \\ 0_{n\cdot(2m+4l+4)n} & & I_n \end{bmatrix}, \qquad \Upsilon_2 = \begin{bmatrix} I_{mn} & & * \\ 0_{mn\cdot n} & I_{mn} & * \\ 0_{ln\cdot mn} & I_{ln} & * \\ 0_{ln\cdot(m+1)n} & I_{ln} & * \\ 0_{mn\cdot(m+l+1)n} & I_{mn} & * \\ 0_{mn\cdot(m+l+2)n} & I_{mn} & * \\ 0_{ln\cdot(2m+l+1)n} & I_{ln} & * \\ 0_{ln\cdot(2m+l+2)n} & I_{ln} & * \\ 0_{ln\cdot(2m+2l+2)n} & I_{ln} & * \\ 0_{ln\cdot(2m+3l+2)n} & I_{ln} & * \end{bmatrix}$$

with all $*$ above representing the appropriately dimensional $\mathbf{0}$ matrix making $I_{1j}, I_{2i}, \Upsilon_1, \Upsilon_2$ of π columns, and

$$\Theta_{11} = -E_1^T C - C^T E_1 - U_1\Sigma_1, \qquad \Theta_{14} = E_1^T A + U_1\Sigma_2, \qquad \Theta_{17} = P - \Sigma K + \bar{\Sigma} F - E_1^T - C^T E_2,$$

$$\Theta_{47} = K - F + A^T E_2, \qquad \Theta_{77} = -E_2^T - E_2 + \sum_{i=1}^{m} \varrho V_i + \sum_{i=1}^{l} \sum_{j=i}^{l} \delta W_{ij},$$

$$\Xi_{99} = \left(1 - \frac{\mu_m}{l}\right)\bar{X}_2 - \left(1 - \frac{\mu_0}{l}\right)\tilde{X}_1 - \tilde{R}\tilde{\Sigma}_1,$$

$$\Xi_{9,10} = \left(1 - \frac{\mu_m}{l}\right)\bar{Y}_2 - \left(1 - \frac{\mu_0}{l}\right)\tilde{Y}_1 - \tilde{R}\tilde{\Sigma}_2, \qquad \Xi_{10,10} = \left(1 - \frac{\mu_m}{l}\right)\bar{Z}_2 - \left(1 - \frac{\mu_0}{l}\right)\tilde{Z}_1 - \tilde{R}.$$

Remark 1. Presently, the convex combination technique has been widely employed owing to the fact that it could help reduce the conservatism more effectively than the previous ones; one can see [25,14–19,27,28]. Yet, those works have always ignored the lower bound on $\dot{\tau}(t)$, which was fully considered in this work. Thus our methods can be more applicable than the present ones when the lower bound of time derivative is available. Though the conditions are not presented in the forms of standard LMIs, it is still straightforward and convenient to check the feasibility without tuning any parameters by utilizing LMI in Matlab Toolbox.

Remark 2. As for $V_2(x(t))$ and $1 \le i \le j \le l$ in (7), if we denote $\begin{bmatrix} X_{2ij} & Y_{2ij} \\ * & Z_{2ij} \end{bmatrix} = 0$ (respectively, $\begin{bmatrix} X_{1ij} & Y_{1ij} \\ * & Z_{1j} \end{bmatrix} = 0$), our result can be true as only μ_m (respectively, μ_0) is available. If we set $\begin{bmatrix} X_{hij} & Y_{hij} \\ * & Z_{hij} \end{bmatrix} = 0$ $(h = 1, 2)$ in (7) simultaneously, Theorems 1 and 2 still hold when μ_0, μ_m are unknown, or $\tau(t)$ is not differentiable.

Remark 3. In view of delay-partitioning idea employed in this work, with integers m, l increasing, the dimension of the derived LMIs will become higher and it will take more computing time to check the stability criteria. Yet, if the lower bound of $\tau(t)$ is given and $l \ge 5$, the maximum allowable delay upper bound $\tau_{\max}$ will become inapparently larger and approach an approximate upper bound [11,12,26,13]. Thus if we want to employ the idea to real cases, we do not necessarily partition the interval $[\tau_0, \tau_m]$ into a large number of subintervals.

Remark 4. It is possible to extend our main results to more complex delayed neural networks, such as delayed Cohen–Grossberg network model [8,15,18], delayed neutral systems [22–24], or DNNs with distributed delay [9,16,18], uncertain parameters [1,2,5,7,8], stochastic perturbations [16], and Markovian jumping parameters.

4. Numerical examples

In this section, three numerical examples will be given to illustrate the derived results. Firstly, we will utilize one example to illustrate the significance of considering the lower bound on delay derivative.

Example 1. Consider the delayed neural networks (21) with the following parameters:

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{\Sigma} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.8 \end{bmatrix}$$

which has been addressed in [10,19]. If we set $\tau_0 = 0$ and do not consider the existence of μ_0, then by utilizing Theorem 2 and Remark 2, the corresponding maximum allowable upper bounds (MAUBs) $\tau_{\max}$ for different μ_m derived by the results in [19] and in the paper can be summarized in Table 1, which demonstrates that Theorem 1 of $l = 1$ is more conservative than the one in [19].

Ref. [19] provides one piecewise delay method to study the asymptotic stability for DNNs, in which the interval of variable delay $\tau(t)$ is divided into two segments by using its central point. Then $\tau_{\max}$ in [19] was derived based on the method of dividing delay interval $[\tau_0, \tau_m]$ into two subintervals. Thus it might be reasonable that the $\tau_{\max}$ of $l = 1$ in our paper is smaller than the relevant one in [19]. Yet, if we set $\mu_0 = 0.5$, it is easy to verify that our results can yield much less conservative results than the one in [19], which can be shown in the following table.

Based on Tables 1 and 2, it indicates that the conservatism of stability criterion can be greatly deduced if we take the lower bound of delay derivative into consideration. Moreover, though the delay-partitioning idea has been used in [10], the corresponding MAUBs $\tau_{\max}$ derived by [10] and Theorem 1 are summarized in the following table, which shows that our delay-partitioning idea can be more efficient than the one in [10] even for $l = 1, 2$ (Table 3).

Example 2. We consider the delayed neural networks (21) with

$$C = \mathrm{diag}\{1.2769, 0.6231, 0.9230, 0.4480\} \qquad \Sigma = 0_{3\times 3} \qquad \bar{\Sigma} = \mathrm{diag}\{0.1137, 0.1279, 0.7994, 0.2368\}$$

$$A = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix} \qquad B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}$$

which has been addressed extensively; see [13,14] and the references therein. Together with the delay-partitioning idea and for different μ_m, the works [13,14] have calculated the MAUBs $\tau_{\max}$ such that the origin of the system is globally asymptotically stable for $\tau(t)$ satisfying $3 = \tau_0 \le \tau(t) \le \tau_m \le \tau_{\max}$. By resorting to Theorem 2 and Remark 2, the

Table 1
The calculated delay upper bound $\tau_{\max}$ for various l, μ_m, and $\tau_0 = 0$.

Methods \ μ_m	0.6	0.8	0.9	1.2
Zhang in [19]	3.5209	2.8654	1.9508	–
Theorem 1 ($l = 1$)	3.4911	2.8554	1.9287	1.2077
Theorem 1 ($l = 2$)	3.7745	3.2113	2.2172	1.3718

Table 2
The calculated delay upper bound $\tau_{\max}$ for various μ_m, and $l = 1$, $\tau_0 = 0$, $\mu_0 = 0.5$.

Methods \ μ_m	0.6	0.8	0.9	1.2
Zhang in [19]	3.5209	2.8654	1.9508	–
Theorem 1 ($l = 1$)	3.5922	2.8934	1.9923	1.2145

Table 3
The calculated delay upper bound $\tau_{\max}$ for various l, μ_m, and $\tau_0 = 0$, $\mu_0 = 0.5$.

Methods \ μ_m	0.8	0.9	Unknown μ_m
Chen [10] $l = 1$	1.8496	1.1650	1.0904
$l = 2$	1.9149	1.1786	1.0931
Theorem 1 $l = 1$	2.8845	1.9678	1.2102
$l = 2$	3.2148	2.2086	1.4117

Table 4
The calculated delay upper bound $\tau_{\max}$ for various l, μ_m, and $\tau_0 = 3$.

Methods \ μ_m	0.1	0.5	0.9	Unknown μ_m
Zhang [13] $l = 1$	3.28	–	–	–
$l = 2$	3.54	–	–	–
Hu [14] $l = 1$	3.33	3.16	3.10	3.09
$l = 2$	3.65	3.32	3.26	3.24
Theorem 1 $l = 1$	3.35	3.21	3.20	3.19
$l = 2$	3.78	3.45	3.39	3.38

Table 5
The calculated delay upper bound $\tau_{\max}$ for various l, μ_m, and $\tau_0 = 0.5$, $\mu_0 = 0$.

Theorem 1\ μ_m	0.8	0.9	1.2	Unknown μ_m
$m = 1, l = 1$	1.8873	1.3442	1.3325	1.3324
$m = 1, l = 2$	2.1005	1.6473	1.5446	1.5446
$m = 1, l = 3$	2.2114	1.7715	1.6874	1.6873

corresponding results can be given in the following table, which indicates that our delay-partitioning idea can be more effective than the relevant ones in [13,14] for $m = 1$, $l = 1, 2$ and $\mu_0 = 0$ (Table 4).

Example 3. Consider the delayed neural networks (4) with the following parameters:

$$\beta(x) = \begin{bmatrix} 4.2x_1 + 0.2\sin^2 x_1 \\ 3.8x_2 + 0.2\cos^2 x_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -1.66 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ -2.475 & 1 \end{bmatrix},$$
$$f(x) = \begin{bmatrix} 0.3(|x_1+1| - |x_1-1|) \\ 0.3(|x_2+1| - |x_2-1|) \end{bmatrix}.$$

Then, $\Gamma = \begin{bmatrix} 4.0 & 0 \\ 0 & 3.6 \end{bmatrix}$, $\Sigma_1 = \begin{bmatrix} -0.36 & 0 \\ 0 & -0.36 \end{bmatrix}$, and $\Sigma_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If $\tau_0 = 0.5$ and $\mu_0 = 0$ are given, then we can set $m = 1$. The corresponding MAUBs $\tau_{\max}$ for different μ_m and l derived by Theorem 1 and Remark 2 can be summarized in the following table and it can be checked that the conservatism reduction proves to be more evident as the integer l increases (Table 5).

5. Conclusions

This paper has investigated the asymptotical stability for a class of neural networks with interval variable delay. By introducing the improved idea of delay-partitioning and constructing one improved the Lyapunov–Krasovskii functional, two stability criteria with significantly reduced conservatism have been established in terms of LMIs. The proposed stability conditions benefit from the partition of delay intervals and convex combination technique. Three numerical examples have been given to demonstrate the effectiveness of the presented criteria and the improvements over the existent methods.

Finally, it should be worth noting that the delay-partitioning idea presented in this work are widely applicable in many cases.

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